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The three-dimensional contact problem with friction for an elastic wedge☆

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ABSTRACT

Solutions of three-dimensional boundary-value problems of the theory of elasticity are given for a wedge, on one face of which a concentrated shearing force is applied, parallel to its edge, while the other face is stress-free or is in a state of rigid or sliding clamping. The solutions are obtained using the method of integral transformations and the technique of reducing the boundary-value problem of the theory of elasticity to a Hilbert problem, as generalized by Vekua (functional equations with a shift of the argument when there are integral terms). Using these and previously obtained equations, quasi-static contact problems of the motion of a punch with friction at an arbitrary angle to the edge of the wedge are considered. In a similar way the contact area can move to the edge of a tooth in Novikov toothed gears. The method of non-linear boundary integral equations is used to investigate contact problems with an unknown contact area.

The problem of the action on the boundary of a half-space of an arbitrarily directed concentrated force splits into two problems: the Boussinesq problem and the Cerruti problem.[1] Using the corresponding solutions, the motion of a punch with friction on the boundary of a half-space was investgated.[2,3] The similar three-dimensional problem for a wedge can be divided into three problems: the problem of a normal force, the problem of the perpendicular to the edge of the wedge shearing force and the problem of the shearing force parallel to the edge of the wedge. Solutions of the first two problems have already been obtained for a wedge, which enable the motion of a punch with friction perpendicular to the edge of the wedge to be investigated.[4–6] The third problem has only been solved for the case of sliding clamping of the other edge of the wedge.[7] The solution of the third problem, given below for the case of one stress-free face, is identical with the Cerruti solution,[1] when the wedge turns in a half-space.

1. The forces parallel to the edge of the wedge

We will consider, in cylindrical coordinates, a wedge $\{r \in [0, \infty]; \varphi \in [-\alpha, \alpha]; z \in [-\infty, \infty]\}$ with elastic characteristics G (the shear modulus) and ν (Poisson's ratio); the z axis is directed along the edge of the wedge. Suppose the face of the wedge $\varphi = -\alpha$ is stress-free, and a concentrated shearing force T acts parallel to the edge on the face $\varphi = \alpha$ at the point $r = x$, $z = y$. The boundary conditions of the problem have the form ($\delta(x)$ is the Dirac delta function)

$$\begin{aligned} &\varphi = \alpha: \sigma_\varphi = \tau_{r\varphi} = 0, \quad \tau_{\varphi z} = T\delta(r-x)\delta(z-y) \\ &\varphi = -\alpha: \sigma_\varphi = \tau_{r\varphi} = \tau_{\varphi z} = 0 \end{aligned} \tag{1.1}$$

It is assumed that as $\rho = (r^2 + z^2)^{1/2} \to \infty$ the displacements descrease not more slowly than ρ^{-1}, while the stresses decrease not more slowly than ρ^{-2}. Moreover, the behaviour of the stresses σ_r, σ_φ and $\tau_{r\varphi}$ on the edge of the wedge (the order of approach to zero or infinity) should be the same as in the plane problem.[8]

Using three arbitrary Neuber–Papkovich harmonic functions $\Phi_n = \Phi_n(r, \varphi, z)$ ($n = 0,1,2$), we will express the displacements in terms of them (Ref. 4, formulae (1.2)). The formulae for the stresses are obtained using Hooke's law.[9–11]

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To solve problem (1.1) we will use the Fourier method and Kontorovich–Lebedev complex integral transformations, which lead to functional equations with a shift in the argument. These equations can be reduced to Fredholm integral equations of the second kind. Changing to real integral transformations, we can write the solution in the form ($K_{i\tau}(\beta x)$ is the Bessel function)

$$\Phi_n(r,\varphi,z) = \frac{2T}{\pi^3 Gx}\int_0^\infty\int_0^\infty \frac{\sin\beta(z-y)}{\beta}\operatorname{sh}\pi\tau\times$$
$$\times[A_n(\tau,\beta)\operatorname{ch}\varphi\tau + B_n(\tau,\beta)\operatorname{sh}\varphi\tau]K_{i\tau}(\beta r)d\tau d\beta,\quad n = 0, 1, 2 \tag{1.2}$$

where

$$A_0(\tau,\beta) = -\frac{1-2\nu}{2\beta\operatorname{sh}\alpha\tau}I_1(\tau),\quad B_0(\tau,\beta) = \frac{1-2\nu}{2\beta\operatorname{ch}\alpha\tau}I_2(\tau)$$

$$A_1(\tau,\beta) = \frac{Z_2(\tau)c(\tau) - xK'_{i\tau}(\beta x)s(\tau)}{\Delta_+(\tau)},\quad B_1(\tau,\beta) = -\frac{Z_1(\tau)s_*(\tau) + xK'_{i\tau}(\beta x)c_*(\tau)}{\Delta_-(\tau)}$$

$$A_2(\tau,\beta) = -\frac{Z_1(\tau)c_*(\tau) - xK'_{i\tau}(\beta x)s_*(\tau)}{\Delta_-(\tau)},\quad B_2(\tau,\beta) = \frac{Z_2(\tau)s(\tau) + xK'_{i\tau}(\beta x)c(\tau)}{\Delta_+(\tau)}$$

$$I_k(\tau) = \int_0^\infty \frac{W_k(t)\Psi_k(t)}{\operatorname{ch}\pi t + \operatorname{ch}\pi\tau}\operatorname{sh}\frac{\pi t}{2}dt,\quad \Delta_\pm(\tau) = \operatorname{ch}2\alpha\tau \pm \cos 2\alpha$$

$$Z_m(\tau) = \frac{2(1-\nu)W_m(\tau)}{\operatorname{ch}(\pi\tau/2)}\Psi_m(\tau) \pm \frac{2(1-\nu)}{1-2\nu}\tau K_{i\tau}(\beta x) \pm \frac{\tau x K'_{i\tau}(\beta x)}{1-2\nu}$$

$$W_m(\tau) = \pm\frac{\Delta_\mp(\tau)}{\operatorname{sh}2\alpha\tau \pm \tau\sin 2\alpha},\quad \Psi_m(\tau) = \Psi_m^*(\tau) \mp \operatorname{ch}\frac{\pi\tau}{2}\theta_m(\tau)$$

$$c(\tau) = \operatorname{ch}\alpha\tau\cos\alpha,\quad s(\tau) = \operatorname{sh}\alpha\tau\sin\alpha,\quad c_*(\tau) = \operatorname{ch}\alpha\tau\sin\alpha,\quad s_*(\tau) = \operatorname{sh}\alpha\tau\cos\alpha \tag{1.3}$$

The prime denotes a derivative with respect to x; the upper sign (plus/minus) in the relations containing the subscript m, corresponds to m = 1, and the lower sign corresponds to m = 2. The functions $\Psi_m^*(\tau)$ $(m = 1, 2)$ satisfy Fredholm integral equations of the second kind $(0 \le \tau < \infty)$

$$\Psi_m^*(\tau) = (1-2\nu)\int_0^\infty L_m(\tau,u)\left[\Psi_m^*(u) \mp \operatorname{ch}\frac{\pi u}{2}\theta_m(u)\right]du \tag{1.4}$$

where

$$L_m(\tau,u) = 2\operatorname{ch}\frac{\pi\tau}{2}\operatorname{sh}\frac{\pi u}{2}W_m(u)\int_0^\infty \frac{\operatorname{sh}\pi t g_m(t)dt}{(\operatorname{ch}\pi t + \operatorname{ch}\pi\tau)(\operatorname{ch}\pi t + \operatorname{ch}\pi u)}$$

$$g_1(t) = \frac{\operatorname{cth}\alpha t\sin^2 2\alpha}{\operatorname{ch}2\alpha t - \cos 4\alpha},\quad g_2(t) = \frac{\operatorname{th}\alpha t\sin^2 2\alpha}{\operatorname{ch}2\alpha t + \cos 4\alpha}$$

$$\theta_m(\tau) = \frac{K_{i\tau}(\beta x)}{\Delta_\mp(\tau)}\cdot\frac{\tau^2\sin 2\alpha \pm 2(1-\nu)\tau\operatorname{sh}2\alpha\tau}{1-2\nu} +$$
$$+\frac{xK'_{i\tau}(\beta x)}{\Delta_\mp(\tau)}\left[\frac{\tau^2\sin 2\alpha \pm \tau\operatorname{sh}2\alpha\tau}{2(1-\nu)(1-2\nu)} + \sin 2\alpha\right] +$$
$$+\int_0^\infty\left[K_{iy}(\beta x)y\sin 2\alpha \mp \frac{xK'_{iy}(\beta x)}{2-2\nu}(1-2\nu)\operatorname{sh}(2\alpha y \mp y\sin 2\alpha)\right]\frac{\operatorname{sh}\pi y dy}{(\operatorname{ch}\pi y - \operatorname{ch}\pi\tau)\Delta_\mp(y)} \tag{1.5}$$

When one face of the wedge is rigidly clamped, it is more convenient to consider the region $\varphi \in [0, \alpha]$. The solution of the problem with boundary conditions (1.1) when $\varphi = \alpha$ and

$$\varphi = 0{:}\ u_r = u_\varphi = u_z = 0 \tag{1.6}$$

is obtained similarly. The Neuber–Papkovich functions have the form (1.2) where

$$A_0(\tau,\beta) = A_2(\tau,\beta) = 0, \quad B_0(\tau,\beta) = -\frac{2(1-2\nu)}{\beta \operatorname{ch}\alpha\tau} I_3(\tau)$$

$$A_1(\tau,\beta) = -\frac{\tau}{\kappa} B_2(\tau,\beta) + \frac{8(1-\nu)S(\tau)}{k\Delta_+(\tau)}$$

$$B_1(\tau,\beta) = -\frac{2}{(1-2\nu)\Delta(\tau)}\{2(1-\nu)[\tau K_{i\tau}(\beta x) - 2R_1(\tau)](\tau s(\tau) + \kappa c(\tau)) +$$

$$+ xK'_{i\tau}(\beta x)[(\tau^2 + (1-2\nu)\kappa)s(\tau) + 2(1-\nu)\tau c(\tau)] + 4(1-\nu)(1-2\nu)S(\tau)\}$$

$$B_2(\tau,\beta) = -\frac{2\kappa}{(1-2\nu)\Delta(\tau)}\left\{2(1-\nu)[\tau K_{i\tau}(\beta x) - 2R_1(\tau)]c_*(\tau) + \right.$$

$$\left. + xK'_{i\tau}(\beta x)[\tau c_*(\tau) - (1-2\nu)s_*(\tau)] + \frac{4(1-\nu)(1-2\nu)\sin 2\alpha S(\tau)}{k\Delta_+(\tau)}\right\}$$

$$S(\tau) = R_1(\tau)c(\tau) + R_2(\tau)s(\tau), \ R_1(\tau) = -(1-2\nu)\frac{W_3(\tau)\Psi_3(\tau)}{\operatorname{ch}(\pi\tau/2)}, \ R_2(\tau) = \int_0^\infty \frac{R_1(t)\operatorname{sh}\pi t}{\operatorname{ch}\pi t - \operatorname{ch}\pi\tau} dt$$

$$W_3(\tau) = \frac{2\Delta(\tau)}{2\tau^2(1-\cos 2\alpha) + 2\kappa \operatorname{ch} 2\alpha\tau + \kappa^2 + 1}, \quad \Psi_3(\tau) = \Psi_3^*(\tau) - \operatorname{ch}\frac{\pi\tau}{2}\theta_3(\tau)$$

$$\Delta(\tau) = \kappa \operatorname{sh} 2\alpha\tau - \tau\sin 2\alpha, \quad \kappa = 3-4\nu \tag{1.7}$$

The function $\Psi_3^*(\tau)$ satisfies Fredholm integral equation of the second kind (1.4) for m = 3 (the minus sign under the integral is taken) with a kernel defined by the first formula of (1.5) with m = 3, in which

$$g_3(t) = -\frac{\operatorname{th}\alpha t \sin^2 2\alpha}{\operatorname{ch} 2\alpha t + \cos 4\alpha} + \{\sin^2\alpha(f_0(t)[2f_1(t) - tf_2(t)] - f_3(t)[2f_2(t) + tf_1(t)]) -$$

$$-2(1-\nu)\sin\alpha(f_0(t)[\sin 3\alpha - \operatorname{ch} 2\alpha t \sin\alpha] - f_3(t)\operatorname{sh} 2\alpha t\cos\alpha)\}/f_4(t)$$

$$f_0(t) = \kappa\operatorname{sh} 2\alpha t\cos 2\alpha - t\sin 2\alpha, \quad f_1(t) = -\operatorname{ch} 2\alpha t + \cos 2\alpha + \sin^2 2\alpha$$

$$f_2(t) = \operatorname{th}\alpha t\sin 2\alpha(1+\cos 2\alpha), \quad f_3(t) = (\kappa\operatorname{ch} 2\alpha t - 1)\sin 2\alpha$$

$$f_4(t) = [f_0^2(t) + f_3^2(t)](\operatorname{sh}^2\alpha t + \cos^2 2\alpha)$$

$$\theta_3(\tau) = \frac{\tau K_{i\tau}(\beta x)}{\Delta(\tau)} \cdot \frac{\tau^2\sin^2\alpha + 2(1-\nu)\kappa\operatorname{ch}^2\alpha\tau}{1-2\nu} +$$

$$+ \frac{\tau x K'_{i\tau}(\beta x)}{\Delta(\tau)}\left[\frac{\tau^2\sin^2\alpha + \kappa\operatorname{ch}^2\alpha\tau + (1-2\nu)^2}{2(1-\nu)(1-2\nu)} + \sin^2\alpha\right] +$$

$$+ \int_0^\infty \left[K_{iy}(\beta x)y^2\sin^2\alpha - \frac{xK'_{iy}(\beta x)}{2-2\nu}((1-2\nu)\kappa\operatorname{sh}^2\alpha y - y^2\sin^2\alpha)\right]\frac{\operatorname{sh}\pi y\, dy}{(\operatorname{ch}\pi y - \operatorname{ch}\pi\tau)\Delta(y)} \tag{1.8}$$

For sliding clamping of one face of the wedge, we will also consider the region $\varphi \in [0, \alpha]$. The solution of the problem with boundary conditions (1.1) with $\varphi = \alpha$ and

$$\varphi = 0: u_\varphi = \tau_{r\varphi} = \tau_{\varphi z} = 0 \tag{1.9}$$

is given by formulae (1.2), where

$$A_1(\tau,\beta) = B_0(\tau,\beta) = B_2(\tau,\beta) = 0 \tag{1.10}$$

while the functions $A_0(\tau, \beta)$, $A_2(\tau, \beta)$ and $B_1(\tau,\beta)$ are defined by the corresponding formulae (1.3), the right-hand sides of which must be multiplied by 2.[7] By virtue of symmetry, this will also be the solution of the boundary-value problem for the region $\varphi \in [-\alpha, \alpha]$ with boundary conditions

$$\varphi = \pm\alpha: \sigma_\varphi = \tau_{r\varphi} = 0, \quad \tau_{\varphi z} = T\delta(r-x)\delta(z-y) \tag{1.11}$$

For all the problems considered, the solutions of the Fredholm integral equations of the second kind (1.4) (m = 1, 2, 3) in the Banach space $\mathbf{C}_M(0, \infty)$ of continuous functions bounded on the semiaxis can be represented by Neumann functional series in powers of (1-2ν), which uniformly converge for sufficiently small values of (1-2ν). We can use the method of mechanical quadratures to solve Eqs (1.4) numerically. Integral equations (1.4) degenerate into equalities for a half-space, and also when the material of the wedge is incompressible (ν = 0.5).

We verified the existence of the passage to the limit $\nu \to 0.5$ in solutions (1.2) - (1.5) and (1.7), (1.8): terms containing (1-2ν) in the denominator vanish.

We also verified that when $\alpha = \pi/2$, solution (1.2) - (1.5) reduces to the solution of the Cerruti problem for a half-space. For example, taking the quadratures,[12,13] we obtain the well-known result (Ref. 1, the third formula of (9.19) with z = 0)

$$u_\varphi(r, \pi/2, z) = -\frac{T(1-2\nu)(z-y)}{4\pi G R^2}, \quad R = ((r-x)^2 + (z-y)^2)^{1/2} \tag{1.12}$$

(in view of the fact that the direction of the z axis is opposite to the one taken previously, the sign of the right-hand side of equality (1.12) was changed).

We will consider, as an example, the tangential displacement u_z on the edge of a wedge (r = 0) with one stress-free face when ν = 0.5 and y = 0. Taking the equality $K_{i\tau}(0) = \delta(\tau)$ into account and evaluating the integral (Ref. 13, formula 2.16.14.4) we obtain

$$u_z(0, \alpha, z) = \frac{T}{8G\alpha}\left(\frac{1}{R_0} + \frac{z^2}{R_0^3}\right), \quad R_0 = (x^2 + z^2)^{1/2} \tag{1.13}$$

When $\alpha = \pi/2$ expression (1.13) is identical with the known expression (Ref. 1, the first formula of (9.19) with ν = 0.5). For any wedge angle the displacement (1.13) reaches a maximum when $z = \sqrt{2}x/2$.

2. The motion of a punch with friction

We will consider the quasi-static contact problem, when a rigid punch begins to move (without sag) over the face $\varphi = \alpha$ of a three-dimensional elastic wedge in an arbitrary direction, making an angle β_0 ($0 \le \beta_0 \le \pi/2$) with the positive direction of the z axis (with the edge of the wedge). The Coulomb friction forces are in the opposite direction to the direction of motion. The face of the wedge $\varphi = -\alpha$ is stress-free (Problem A), or the face of the wedge $\varphi = 0$ is under conditions of rigid or sliding clamping (Problems B and C respectively). The shape of the base of the punch base is described by the function

$$f(r, z) = \frac{(r-a)^2}{2R_1} + \frac{z^2}{2R_2} \tag{2.1}$$

A normal indenting force P having branches H_r (with respect to the semiaxis r) and H_z (with respect to the edge of the wedge) acts on the punch, and also a shearing force T, in the direction of motion. By Coulomb's law the relation $T = \mu P$ is satisfied, where μ is the coefficient of friction. The condition for contact between the punch and the wedge has the form

$$u_\varphi(r, \alpha, z) = -d(r, z) = -[d - f(r, z)], \quad (r, z) \in \Omega \tag{2.2}$$

where $u_\varphi(r, \alpha, z)$ is the normal displacement of the wedge when normal and shearing stresses act on its face, d is the indentation of the punch, and Ω is the unknown contact area. When $\beta_0 = 0$ the punch moves parallel to the edge of the wedge; when $\beta \neq 0$ the punch moves away from the edge when $\mu > 0$ and approaches the edge when $\mu < 0$.

For known values of α, β_0, μ, G, ν, d and of the specified function f(r,z), it is required to determine the contact pressure $\sigma_\varphi(r, \alpha, z) = -q(r, z)$, $(r, z) \in \Omega$, and also the contact area Ω. By finding q(r,z) and Ω, we can obtain the quantities P, H_r and H_z from the punch equilibrium conditions

$$P = \iint_\Omega q(r, z)drdz, \quad PH_r = \iint_\Omega q(r, z)zdrdz, \quad PH_z = \iint_\Omega q(r, z)rdrdz \tag{2.3}$$

Using the solutions of the problems of the action of a shearing force parallel to the edge obtained above, the known solutions of the problems of the action of a normal force and of a shearing force perpendicular to the edge,[4–6,10] and also conditions (2.2), we can derive the following integral equation for the function q(r,z)

$$\begin{gathered}\iint_\Omega q(x, y)K(x, y, r, z)dxdy = 2\pi\theta d(r, z), \quad (r, z) \in \Omega, \quad \theta = G/(1-\nu) \\ K(x, y, r, z) = K_1(x, y, r, z) + \mu\cos\beta_0 K_2(x, y, r, z) + \mu\sin\beta_0 K_3(x, y, r, z)\end{gathered} \tag{2.4}$$

Here ($K_n = K_n(x, y, r, z)$, $n = 1, 2, 3$, $U(\tau, u) = \mathrm{ch}\pi\tau\mathrm{c}\pi u - 1$)

$$K_1 = \frac{1}{R} + \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \cos\beta(z-y)\,\mathrm{sh}\pi\tau\left[(W(\tau) - \mathrm{cth}\pi\tau)K_{i\tau}(\beta x) + \frac{Q_1(\tau)}{\mathrm{ch}(\pi\tau/2)}\right]K_{i\tau}(\beta r)d\tau d\beta$$

$$K_2 = \frac{(1-2\nu)(z-y)}{2(1-\nu)R^2} - \frac{8}{\pi^2 x}\int_0^\infty\int_0^\infty \frac{\sin\beta(z-y)}{\beta}\mathrm{sh}\frac{\pi\tau}{2}Q_2(\tau)K_{i\tau}(\beta r)d\tau d\beta - \frac{2(1-2\nu)}{\pi^2(1-\nu)}\times$$

$$\times\int_0^\infty \frac{\sin\beta(z-y)}{\beta}d\beta\int_0^\infty\int_0^\infty [\mathrm{sh}\pi\tau\,\mathrm{sh}\pi u\,U_1(\tau, u) - U(\tau, u)]\frac{K'_{iu}(\beta x)K_{i\tau}(\beta r)}{\mathrm{ch}\pi u - \mathrm{ch}\pi\tau}du d\tau$$

$$K_3 = \frac{(1-2\nu)(r-x)}{2(1-\nu)R^2} - \frac{8}{\pi^2}\int_0^\infty\int_0^\infty \cos\beta(z-y)\,\mathrm{sh}\frac{\pi\tau}{2}Q_3(\tau)K_{i\tau}(\beta r)d\tau d\beta - \frac{2(1-2\nu)}{\pi^2(1-\nu)}\times$$

$$\times\int_0^\infty \cos\beta(z-y)d\beta\int_0^\infty\int_0^\infty [\mathrm{sh}\pi\tau\,\mathrm{sh}\pi u\,U_2(\tau, u) - U(\tau, u)]\frac{K_{iu}(\beta x)K_{i\tau}(\beta r)}{\mathrm{ch}\pi u - \mathrm{ch}\pi\tau}du d\tau \tag{2.5}$$

For problem A in expression (2.5), we must take

$$W(\tau) = \frac{W_1(\tau) - W_2(\tau)}{2}, \quad Q_1(\tau) = \frac{W_1(\tau)\Phi_1(\tau) - W_2(\tau)\Phi_2(\tau)}{2}$$

$$Q_2(\tau) = \frac{W_1(\tau)S_1(\tau) - W_2(\tau)S_2(\tau)}{2}, \quad U_1(\tau, u) = \frac{1}{2}\left(\frac{W_1(\tau)}{W_1(u)} + \frac{W_2(\tau)}{W_2(u)}\right)$$

$$Q_3(\tau) = \frac{W_1(\tau)\Phi_1^*(\tau) - W_2(\tau)\Phi_2^*(\tau)}{2} - \mathrm{ch}\frac{\pi\tau}{2}f^*(\tau)K_{i\tau}(\beta x)$$

$$U_2(\tau, u) = \frac{W_1(\tau)h_1^*(u) - W_2(\tau)h_2^*(u)}{2(1-2\nu)}, \quad S_m(\tau) = \Psi_m^*(\tau) \mp \mathrm{ch}\frac{\pi\tau}{2}\theta_m^*(\tau), \quad h_m(\tau) = \frac{\tau\sin 2a}{\Delta_\mp(\tau)}$$

$$\theta_m^*(\tau) = -\tau h_m(\tau)K_{i\tau}(\beta x) + \frac{h_m(\tau)}{\tau}xK'_{i\tau}(\beta x) + \int_0^\infty \frac{h_m(u)[K_{iu}(\beta x) + xK'_{iu}(\beta x)]}{\mathrm{ch}\pi u - \mathrm{ch}\pi\tau}\mathrm{sh}\pi u\, du$$

$$f^*(\tau) = \frac{\tau\sin^2 2\alpha}{\tau^2\sin^2 2\alpha - \mathrm{sh}^2 2\alpha\tau}, \quad h_m^*(\tau) = \frac{(1-2\nu)\mathrm{sh}2\alpha\tau \mp \tau\sin 2\alpha}{\Delta_\mp(\tau)} \tag{2.6}$$

The function $\Phi_m(\tau)$ ($m = 1, 2$) in relations (2.6) satisfy a Fredholm integral equation of the second kind ($0 \le \tau < \infty$)

$$\Phi_m(\tau) = (1-2\nu)\int_0^\infty L_m(\tau, u)\left[\Phi_m(u) + \mathrm{ch}\frac{\pi u}{2}K_{iu}(\beta x)\right]du \tag{2.7}$$

the functions $\Phi_m^*(\tau)$ ($m = 1, 2$) satisfy the Fredholm equations ($0 \le \tau < \infty$)

$$\Phi_m^*(\tau) = (1-2\nu)\int_0^\infty L_m(\tau, u)\left[\Phi_m^*(u) + \mathrm{ch}\frac{\pi u}{2}\chi_m(u)\right]du$$

$$\chi_m(\tau) = -\frac{g_m^*(\tau)K_{i\tau}(\beta x)}{2(1-\nu)(1-2\nu)} + \frac{1}{2(1-\nu)}\int_0^\infty \frac{h_m^*(u)K_{iu}(\beta x)}{\mathrm{ch}\pi u - \mathrm{ch}\pi\tau}\mathrm{sh}\pi u du$$

$$g_m^*(\tau) = \frac{\pm\tau}{W_m(\tau)} \pm \frac{2(1-\nu)(1-2\nu)\sin 2\alpha}{\Delta_\mp(\tau)} \tag{2.8}$$

and the functions Ψ_m^* ($m = 1, 2$) in (2.6) are found from Fredholm equations (1.4).

For Problems B ($m = 3$, $k = 0$) and C ($m = k = 1$), in formulae (2.5) we must put

$$W(\tau) = W_m(\tau), \quad Q_1(\tau) = W_m(\tau)\Phi_m(\tau), \quad Q_2(\tau) = W_m(\tau)S_m(\tau)$$

$$U_1(\tau,u) = \frac{W_m(\tau)}{W_k(u)}, \quad W_0(\tau) = \frac{\Delta(\tau)}{2\tau^2\sin^2\alpha + 2\kappa\,\mathrm{sh}^2\alpha\tau}$$

$$Q_3(\tau) = W_m(\tau)\Phi_m^*(\tau) - \mathrm{ch}\frac{\pi\tau}{2}f_m^*(\tau)K_{i\tau}(\beta x), \quad U_2(\tau,u) = \frac{W_m(\tau)h_m^*(u)}{1-2\nu}$$

$$S_m(\tau) = (2-k)\Psi_m^*(\tau) - \mathrm{ch}\frac{\pi\tau}{2}\theta_m^*(\tau), \quad h_3(\tau) = \frac{2\tau^2\sin^2\alpha}{\Delta(\tau)}$$

$$\theta_3^*(\tau) = -\tau h_3(\tau)K_{i\tau}(\beta x)\left[1 + \frac{2(1-\nu)(1-2\nu)}{\tau^2\sin^2\alpha}\right] +$$

$$+\frac{h_3(\tau)}{\tau}xK'_{i\tau}(\beta x) + \int_0^\infty \frac{h_3(u)[K_{iu}(\beta x) + xK'_{iu}(\beta x)]}{\mathrm{ch}\pi u - \mathrm{ch}\pi\tau}\mathrm{sh}\pi u\,du$$

$$f_1^*(\tau) = \frac{\sin 2\alpha}{\mathrm{sh}2\alpha\tau + \tau\sin 2\alpha}, \quad f_3^*(\tau) = \frac{4\tau\sin^2\alpha}{2\tau^2(1-\cos 2\alpha) + 2\kappa\,\mathrm{ch}2\alpha\tau + \kappa^2 + 1}$$

$$h_3^*(\tau) = \frac{2\kappa(1-2\nu)\mathrm{sh}^2\alpha\tau - 2\tau^2\sin^2\alpha}{\Delta(\tau)}, \quad g_3^*(\tau) = \frac{\tau}{W_3(\tau)} + \frac{4(1-\nu)(1-2\nu)\tau\sin^2\alpha}{\Delta(\tau)} \tag{2.9}$$

The functions $\Phi_m(\tau)\,(m=1{,}3)$ in relations (2.9) satisfy Fredholm equations (2.7), the functions $\Phi_m^*(\tau)\,(m=1,3)$ satisfy Eqs (2.8), and the functions $\Psi_m^*(\tau)\,(m=1.3)$ satisfy Eqs (1.4), in which we must choose the minus sign under the integral.

We have separated the principal terms, corresponding to the elastic half-space,[2] in the kernels (2.5).

To solve integral equation (2.4) numerically with an unknown contact area, we will use the method of Hammerstein-type non-linear boundary integral equations.[4–6,10,14] We will introduce the following notation

$$M = (r,z), \quad N = (x,y) \tag{2.10}$$

and assume that the contact area as a whole is contained within the rectangle

$$S = \{|r-a| \le b, |z| \le c\}, \quad a > b \ge c \tag{2.11}$$

which does not reach the edge of the wedge.

We will supplement Eq. (2.4) with the condition that the contact pressure in the contact area is non-negative, and also the conditions for there to be no contact and for the pressure in the additional region S\W to vanish, writing them all in the form of the system

$$\int_S K(N,M)q(N)dN = 2\pi\theta d(M), \quad q(M) \ge 0, \quad M \in \Omega$$

$$\int_S K(N,M)q(N)dN < 2\pi\theta d(M), \quad q(M) = 0, \quad M \in S\backslash\Omega \tag{2.12}$$

System (2.12) reduces to a single non-linear equation, to solve which we used the modified Newton's method.

For a numerical analysis we chose an incompressible material ($\nu = 0.5$), when the kernel of (2.5) can be simplified considerably. Close to the edge of the elastic quarter-space a considerable reduction in the contact area and contact pressures are observed for Problem A compared with Problem B. When the punch moves parallel to the edge of the quarter-space the effect of friction on the pressure distribution is negligible. When the punch approaches the edge of the quarter-space ($\mu < 0$) for Problem A the pressure and the contact area decrease compared with the analogous case of withdrawal from the edge ($\mu > 0$). For problem B, on the other hand, it is more difficult for the punch to penetrate when $\mu < 0$, than when $\mu > 0$. The least pressure in the case of Problem A occurs when $\beta_0 = \pi/2$ and $\mu < 0$ (approach perpendicular to the edge). Problem C for a wedge with an aperture angle α is equivalent to the symmetrical contact problem of the motion of two punches on the faces of a wedge with an aperture angle 2α. When $\alpha = \pi/3$ in this problem the contact area and the pressure are greater when the punches approach ($\mu < 0$), than in the case when they move away from the edge ($\mu > 0$). Here the greatest contact pressures occur when $\beta_0 = \pi/2$ (Table 1).

Values of the dimensionless force $P/(2\pi\theta b^2)$ as a function of the indentation of the punch d/b, the coefficient of friction μ and the angle of motion β_0 ($b/(2R_1) = b/(2R_2) = c/b = 1$, $a/b = 1.2$), the aperture angle of the wedge is $\pi/3$) are given in the table for Problems A, B and C. The force increases as the indentation increases and depends considerably on the direction of the punch motion. It is easiest of all to impress the punch in the case of Problem A when approaching the edge at a right angle. Unlike the contact problem for an incompressible elastic half-space, where Coulomb friction has no effect on the contact pressure distribution[2] (the displacement (1.12) vanishes), for an incompressible wedge in the case when there is contact with the punch, as the analysis shows, we must take into account the contribution of the friction forces to the value of the indenting force and the pressures in the contact area, close to the edge.

Table 1

μ	β_0	$d/b = 1$			$d/b = 1.5$		
		A	B	C	A	B	C
0.2	0	0.0922	0.586	0.283	0.152	1.33	0.514
	$\pi/6$	0.0969	0.557	0.275	0.160	1.24	0.497
	$\pi/3$	0.101	0.537	0.270	0.167	1.19	0.485
	$\pi/2$	0.102	0.530	0.268	0.169	1.17	0.481
	$\pi/6$	0.0841	0.655	0.302	0.139	1.54	0.554
−0.2	$\pi/3$	0.0851	0.645	0.299	0.140	1.51	0.548
	$\pi/2$	0.0880	0.618	0.292	0.145	1.43	0.533

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